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Partial Rank Symmetry of Distributive Lattices for Fences

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Abstract. Associated with any composition $\beta = (a, b, \ldots)$ is a corresponding fence poset $F(\beta)$ whose covering relations are

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots.$$

The distributive lattice $L(\beta)$ of all lower order ideals of $F(\beta)$ is important in the theory of cluster algebras. In addition, its rank generating function $r(q;\beta)$ is used to define q-analogues of rational numbers. Kantarcı Oğuz and Ravichandran recently showed that its coefficients satisfy an interlacing condition, proving a conjecture of McConville, Smyth, and Sagan, which in turn implies a previous conjecture of Morier-Genoud and Ovsienko that $r(q;\beta)$ is unimodal. We show that, when β has an odd number of parts, then the polynomial is also partially symmetric: the number of ideals of $F(\beta)$ of size k equals the number of filters of size k, when k is below a certain value. Our proof is completely bijective. Kantarcı Oğuz and Ravichandran also introduced a circular version of fences and proved, using algebraic techniques, that the distributive lattice for such a poset is rank symmetric. We give a bijective proof of this result, as well. We end with some questions and conjectures raised by this work.

Mathematics Subject Classification. Primary 06A07; Secondary 05A15, 05A19, 05A20, 06D05.

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1. Introduction

We will be studying the rank sequences for distributive lattices of certain partially ordered sets (posets) called fences, defined as follows. Any terms or notation from the theory of posets which are not defined here can be found in the texts of Sagan [16] or Stanley [22]. A *chain* of length l is a totally ordered set with $l+1$ elements. A *composition of* m is a sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ of

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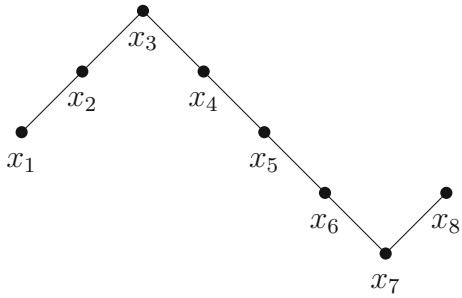

FIGURE 1. The fence $F(2,4,1)$

positive integers, called *parts*, with $\sum_i \beta_i = m$. In this setting, we write $\beta \models m$. The corresponding fence $F(\beta)$ is obtained by taking chains S_i of length β_i for $1 \le i \le s$ and identifying the maximal (respectively, minimal) elements of S_i and S_{i+1} for i odd (respectively, even). As an example, the fence $F(2,4,1)$ is displayed in Fig. 1. Placing the chains $S_1, S_2, \ldots, S_s$ from left to right as in the figure, we label the elements of $F(\beta)$ as $x_1, x_2, \ldots, x_n$ from left to right. We say that S_i is an *ascending* or *descending* segment of $F(\beta)$ depending on whether i is odd or even, respectively. Note that if $\#F(\beta) = n$, where the hash symbol denotes cardinality, then $\beta \models n-1$.

Let $L(\beta)$ be the distributive lattice of lower order ideals of $F(\beta)$. These lattices can be used to compute mutations in an associated cluster algebra on a surface with marked points. In fact there are (at least) six methods for doing so, see [7,14,17,18,20,25,26]. Since $L(\beta)$ is ranked, it has an associated *rank sequence* $r(\beta) : r_0, r_1, \ldots, r_n$ where

$$r_k = \text{number of elements at rank } k \text{ in } L(\beta).$$

for $0 \le k \le n$. The corresponding rank generating functions

$$r(q;\beta) = \sum_{k=0}^{n} r_k q^k$$

were used by Morier-Genoud and Ovsienko to define q-analogues of rational, and even real, numbers [12]. For example, for the fence in Fig. 1, the rank generating function is

$$r(q;(2,4,1)) = 1 + 2q + 4q^2 + 5q^3 + 6q^4 + 5q^5 + 3q^6 + 2q^7 + q^8,$$

and for the fence $F(\beta)$ with $\beta = (6,2,1,2,3,1,6)$ (see Fig. 4), it is

$$\begin{aligned} r(q;\beta) = {} & 1 + 4q + 11q^2 + 23q^3 + 41q^4 + 65q^5 + 94q^6 + 125q^7 + 155q^8 + 181q^9 \\ & + 198q^{10} + 205q^{11} + 200q^{12} + 182q^{13} + 156q^{14} + 125q^{15} + 94q^{16} \\ & + 65q^{17} + 41q^{18} + 23q^{19} + 11q^{20} + 4q^{21} + q^{22}. \end{aligned}$$

Two well-studied properties of sequences $b : b_0, b_1, \ldots, b_n$ are as follows. Call the sequence *symmetric* if

$$b_k = b_{n-k}$$

for $0 \leq k \leq n$. The sequence is said to be *unimodal* if there is an index m, such that

$$b_0 \leq b_1 \leq \cdots \leq b_m \geq b_{m+1} \geq \cdots \geq b_n.$$

Sequences satisfying these properties abound in combinatorics, algebra, and geometry. See the survey articles of Stanley [21], Brenti [5], or Brändén [4] for examples. In their previously cited paper, Morier-Genoud and Ovsienko made the following conjecture which has now been proved, as we will discuss shortly.

Conjecture 1.1. [12]. *For all β, the sequence $r(\beta)$ is unimodal.*

It is not true that $r(\beta)$ is always symmetric. For example, when $\beta = (1, 1)$, we have $r(\beta) : 1, 2, 1, 1$. However, there are other recently studied properties of sequences [1–3,6,19,24] which are satisfied by $r(\beta)$. Call a sequence $b : b_0, b_1, \ldots, b_n$ *top heavy* if

$$b_k \leq b_{n-k}$$

for $0 \leq k < \lfloor n/2 \rfloor$, where $\lfloor \cdot \rfloor$ is the floor (round-down) function. Dually, the sequence is *bottom heavy* if

$$b_k \geq b_{n-k}$$

for $0 \leq k < \lfloor n/2 \rfloor$. Call the sequence *top interlacing* if

$$b_0 \leq b_n \leq b_1 \leq b_{n-1} \leq \cdots \leq b_{\lceil n/2 \rceil},$$

where $\lceil \cdot \rceil$ is the ceiling (round-up) function. Top interlacing clearly implies top heavy, and it also gives unimodality, since the inequalities imply that the sequence is increasing up to $b_{\lceil n/2 \rceil}$ and decreasing from $b_{\lceil n/2 \rceil}$ onward. Some papers use the term "alternately increasing," but we prefer "top interlacing", because it emphasizes how the first and second halves of the sequence interlace. Similarly, define a sequence to be *bottom interlacing* if

$$b_n \leq b_0 \leq b_{n-1} \leq b_1 \leq \cdots \leq b_{\lfloor n/2 \rfloor}.$$

As before, bottom interlacing implies both bottom heavy and unimodal. McConville et al. [13] conjectured the following strengthening of Conjecture 1.1, which has recently been proved by Kantarcı Oğuz and Ravichandran [11] using induction and algebraic techniques.

Theorem 1.2. [11]. *Let $\beta = (\beta_1, \ldots, \beta_s)$.*

(a) *If $s = 1$, then $r(\beta) = (1, 1, \ldots, 1)$.*
(b) *If s is even, then $r(\beta)$ is bottom interlacing.*
(c) *Suppose $s \geq 3$ is odd and let $\beta' = (\beta_2, \ldots, \beta_{s-1})$.*
  (i) *If $\beta_1 > \beta_s$, then $r(\beta)$ is bottom interlacing.*
  (ii) *If $\beta_1 < \beta_s$, then $r(\beta)$ is top interlacing.*
  (iii) *If $\beta_1 = \beta_s$, then $r(\beta)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\beta')$ is symmetric, top interlacing, or bottom interlacing, respectively.*

The purpose of the present work is to show that, even though $r(\beta)$ is not always symmetric, it exhibits at least partial symmetry if there is an odd number of segments. In particular, our main result is as follows.

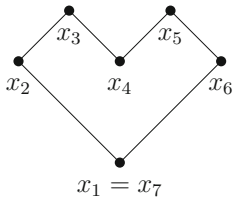

FIGURE 2. The circular fence $\overline{F}(2,1,1,2)$

Theorem 1.3. *Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ where s is odd and $r(\beta) : r_0, r_1, \ldots, r_n$. For all $k \leq \min\{\beta_1, \beta_s\}$, we have*

$$r_k = r_{n-k}.$$

Kantarcı Oğuz and Ravichandran's proof of Theorem 1.2 relied on certain posets obtained by making the Hasse diagram of a fence into a cycle. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell}) \models n$ be a composition with an even number of parts, so that the fence $F(\beta)$ has $n+1$ elements $x_1, \ldots, x_{n+1}$, begins with an ascending segment, and ends with a descending segment. Define the corresponding *circular fence* to be the poset $\overline{F}(\beta)$ with n elements obtained by identifying x_1 and x_{n+1}. For example, $\overline{F}(2,1,1,2)$ is displayed in Fig. 2. Denote the rank sequence of the lattice of lower order ideals of $\overline{F}(\beta)$ by $\overline{r}(\beta)$. Using algebraic manipulation of recurrence relations, Kantarcı Oğuz and Ravichandran proved the following result, and left finding a bijective proof as an open problem.

Theorem 1.4. [11]. *Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ where s is even. Then, $\overline{r}(\beta)$ is symmetric.*

The rest of this paper will be structured as follows. In the next section, we will present a totally bijective proof of Theorem 1.3. Section 3 will be devoted to showing that our bijection can be used, with minor modifications, to prove Theorem 1.4 as well. We will end with a section of comments and open questions.

2. Proof of Partial Symmetry for Fences

To give our bijective proof of Theorem 1.3, we will need some definitions and notation. In a poset, an *ideal* will always be a lower order ideal. We will also use the terms *upper order ideal* and *filter* interchangeably. Consider a composition $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell+1})$ with an odd number of parts. For a fence $F(\beta)$ and $k \geq 0$, we let

$$\mathcal{I}_k(\beta) = \{I \mid I \text{is a lower order ideal of } F(\beta) \text{ with } \#I = k\}$$

and

$$\mathcal{U}_k(\beta) = \{U \mid U \text{ is an upper order ideal of } F(\beta) \text{ with } \#U = k\}.$$

To prove Theorem 1.3, it suffices to construct a bijection $\Phi : \mathcal{I}_k(\beta) \to \mathcal{U}_k(\beta)$ for all $k \leq \min\{\beta_1, \beta_{2\ell+1}\}$. This is because, with the notation of the theorem, we have $\#\mathcal{I}_k(\beta) = r_k$ and $\#\mathcal{U}_k(\beta) = r_{n-k}$.

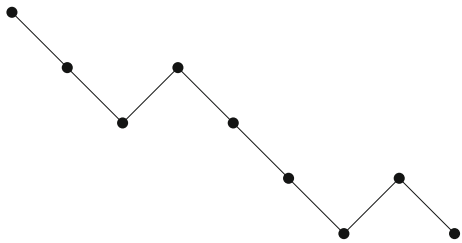

FIGURE 3. The gate $G(2,3,1)$

2.1. Bijection ϕ for Gates

To define Φ, we will first construct a bijective map ϕ on certain ideals of a particular subposet of a fence obtained by removing the first and last segments, and requiring ascending segments to have length one. For an arbitrary composition $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$, let the corresponding *gate* be

$$G(\delta) = F(\delta_1, 1, \delta_2, 1, \ldots, \delta_{\ell-1}, 1, \delta_\ell)^*,$$

where the star indicates poset dual. The gate $G(2,3,1)$ is shown in Fig. 3. We will use the same terminology for gates as we do for fences. Note that $G(\delta)$ begins and ends with a descending segment. Let D_i denote the ith descending segment from the left, which has length δ_i. The ideals of a gate which correspond to those of bounded size in the corresponding fence are as follows. If $G(\delta)$ has ℓ descending segments, then call an ideal I of this gate *restricted* if $\#(I \cap D_1) \leq \delta_1$ and $\#(I \cap D_\ell) \neq 1$. In other words, I is restricted if it does not contain the maximal element on D_1, and if it contains the minimal element on D_ℓ, then it also contains the element above it. Let

$$\mathcal{I}^r(\delta) = \{I \mid I \text{ is a restricted ideal of the gate } G(\delta)\}.$$

Call a filter U of $G(\delta)$ *restricted* if $\#(U \cap D_1) \neq 1$ and $\#(U \cap D_\ell) \leq \delta_\ell$. Equivalently, U^* is a restricted ideal of $G(\delta)^*$, which is isomorphic to $G(\delta^R)$ where

$$\delta^R = (\delta_\ell, \delta_{\ell-1}, \ldots, \delta_2, \delta_1)$$

is the reversal of δ. In general, the reversal of any sequence b will be denoted by b^R. Note the difference between our use of r for restricted and R for reversal. The notation for restricted filters is as expected

$$\mathcal{U}^r(\delta) = \{U \mid U \text{ is a restricted filter of the gate } G(\delta)\}.$$

We will describe a cardinality-preserving bijection

$$\phi : \mathcal{I}^r(\delta) \to \mathcal{U}^r(\delta).$$

We will need some more notation and terminology. Given a sequence $d : d_1, d_2, \ldots, d_\ell$, we use the floor symbol

$$\lfloor d \rfloor = \lfloor d_1, d_2, \ldots, d_\ell \rfloor$$

to denote the subset of $G(\delta)$ (if it exists) consisting of the smallest d_i elements on segment D_i for $1 \leq i \leq \ell$. It is easy to see that $\lfloor d \rfloor$ exists and is a restricted

lower order ideal if and only if the following conditions hold. We use the notation $[m, n]$ for the interval of integers between m and n inclusive, which is shortened to $[n]$ if $m = 1$. The restricted ideal conditions are

I1 (existence) for $i \in [\ell]$, we have $0 \leq d_i \leq \delta_i + 1$,
I2 (lower order ideal) for $i \in [2, \ell]$: if $d_i = \delta_i + 1$, then $d_{i-1} > 0$,
I3 (restricted) $d_1 \leq \delta_1$ and $d_\ell \neq 1$.

Similarly, we use ceiling notation

$$\lceil e \rceil = \lceil e_1, e_2, \ldots, e_\ell \rceil$$

to denote the subset of $G(\delta)$ containing the largest e_i elements on segment i for $1 \leq i \leq \ell$. Here are the conditions for $\lceil e \rceil$ to exist and be a restricted filter:

U1 (existence) for $i \in [\ell]$, we have $0 \leq e_i \leq \delta_i + 1$,
U2 (upper order ideal) for $i \in [\ell - 1]$: if $e_i = \delta_i + 1$, then $e_{i+1} > 0$ for,
U3 (restricted) $e_1 \neq 1$ and $e_\ell \leq \delta_\ell$.

A *factor* of the sequence $d : d_1, d_2, \ldots, d_\ell$ is a subsequence $d_i, d_{i+1}, \ldots, d_j$ of consecutive elements. If the d_i's are nonnegative integers, then a *block* is a maximal factor of positive integers. For example, the sequence

$$d : 6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$

has three blocks, namely, $6, 1, 1, 1$; $4, 5, 1, 1$; and $3, 1, 2$. The *factor of trailing ones* of a block B is the (possibly empty) maximal factor T of B consisting only of ones, such that there is no element of B larger than one to its right. In our example, the blocks have three, two, and no trailing ones, respectively. Note that if $\lfloor d_1, d_2, \ldots, d_\ell \rfloor$ is a restricted ideal, then any nonempty factor T of trailing ones must be followed by a 0. This follows directly from the definition of T unless its block contains the last element d_ℓ. And in that case, since the ideal is restricted, we must have $d_\ell \geq 2$, so that no trailing ones are possible.

One can now construct $\phi(\lfloor d_1, d_2, \ldots, d_\ell \rfloor)$ as follows. Consider each block B of the sequence $d_1, d_2, \ldots, d_\ell$, and factor it as the concatenation $B = B'T$ where T is B's factor of trailing ones and B' is the rest of B. The map ϕ performs the following two steps:

P1 For each nonempty factor T of trailing ones, exchange T with the 0 to its right.
P2 For each B' with $\#B' \geq 2$, decrease the rightmost such entry by 1 and increase the leftmost one by 1.

Intuitively, if any descending segment is empty and any segments immediately to its left contain only single elements of the ideal, then these elements are each pushed to the right by one segment. And if there are any consecutive sequences of nonzero elements which were not pushed right, then the rightmost element of each such sequence is moved to the segment of the leftmost element. (Therefore, if the sequence only consists of one element, then the net effect is no movement at all.) Continuing our example, the three blocks have 3, 2 and 0 trailing ones and B' equal to 6; $4, 5$; and $3, 1, 2$ from left to right. Therefore, after P1, we have the sequence

$$6, 0, 1, 1, 1, 4, 5, 0, 1, 1, 0, 3, 1, 2.$$

Now, applying P2 gives

$$\phi(\lfloor 6,1,1,1,0,4,5,1,1,0,0,3,1,2\rfloor) = \lceil 6,0,1,1,1,5,4,0,1,1,0,4,1,1\rceil.$$

Note that the construction of $\phi(\lfloor d\rfloor)$ does not depend on the lengths δ_i.

For the following proof, it will be convenient to extend the reversal operator as follows. If $\lfloor d\rfloor$ is an ideal of $G(\delta)$, then let

$$\lfloor d\rfloor^R = \lceil d^R\rceil,$$

where $\lceil d^R\rceil$ is being considered as a filter of $G(\delta^R)$. Similarly, let

$$\lceil e\rceil^R = \lfloor e^R\rfloor.$$

Theorem 2.1. *The map $\phi : \mathcal{I}^r(\delta) \to \mathcal{U}^r(\delta)$ defined by* P1 *and* P2 *is a cardinality-preserving bijection.*

Proof. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$, and suppose we are given $d : d_1, d_2, \ldots, d_\ell$ with $\lfloor d\rfloor \in \mathcal{I}^r(\delta)$. Let $\phi(\lfloor d\rfloor) = \lceil e\rceil$, where $e : e_1, e_2, \ldots, e_\ell$.

We first show that ϕ is well defined in that $\#\lfloor d\rfloor = \#\lceil e\rceil$ and $\lceil e\rceil \in \mathcal{U}^r(\delta)$. The first statement is clear, since P1 does not change cardinalities, and every entry increased by one in P2 is offset by an entry decreased by one. For the second statement, we need to check U1–U3. The truth of U1 follows from the fact that d satisfies I1 unless $d_i = \delta_i + 1$ and d_i is increased in step P2. However, if $i = 1$, then this contradicts I3, and if $i > 1$, then this contradicts I2, since d_i was not the first nonzero entry in its block. If U2 is violated, then $e_i = \delta_i + 1$ and $e_{i+1} = 0$. Therefore, d_i must have been the last entry of some B'. If $\#B' = 1$, then $d_{i-1} = 0$. However, then $e_i = d_i \leq \delta_i$, because if we had $d_i = \delta_i + 1$, then $\lfloor d\rfloor$ would not be an ideal, since it violates I2. On the other hand, if $\#B' > 1$, then by P2, we have $e_i = d_i - 1 \leq \delta_i$, which is another contradiction. Thus, $\lceil e\rceil$ is a filter. Finally, we must verify U3. For the first condition, suppose, towards a contradiction, that $e_1 = 1$, and let B' be the initial factor of the block B containing $d_1 \geq 1$. If $e_1 = d_1$, then by P2, we must have $\#B' = 1$. But then, B' would have been included in the trailing ones of B and moved to the right in P1. The other possibility is $e_1 = d_1 + 1 \geq 2$, again a contradiction. Thus, the first condition holds. To prove that the second condition is true, assume the opposite which is that $e_\ell = \delta_\ell + 1$. Clearly, $e_\ell \geq 2$. It follows that d_ℓ must have been part of a block B with no trailing ones, so that $B' = B$. If $\#B' = 1$, then $d_\ell = e_\ell = \delta_\ell + 1$. By P2, this forces $d_{\ell-1} \neq 0$. But then, d_ℓ was not the only element in B' which is impossible. If $\#B' \geq 2$, then, by P2 again, $e_\ell = d_\ell - 1 \leq \delta_\ell$. This final contradiction finishes the proof that U3 holds and that ϕ is well defined.

To show that ϕ is bijective, we construct $\phi^{-1} : \mathcal{U}^r(\delta) \to \mathcal{I}^r(\delta)$. If $\lceil e\rceil \in \mathcal{U}^r(\delta)$, then define

$$\phi^{-1}(\lceil e\rceil) = \phi(\lceil e\rceil^R)^R,$$

where, on the right-hand side, the map being applied is $\phi : \mathcal{I}^r(\delta^R) \to \mathcal{U}^r(\delta^R)$. Because reversal is an involution, showing that ϕ^{-1} is well defined is equivalent to showing that $\lceil e\rceil$ is a restricted filter if and only if $\lceil e\rceil^R$ is a restricted ideal. However, this follows immediately by comparing I1–I3 with U1–U3.

To show that ϕ^{-1} is indeed the inverse of ϕ, we claim that the factors of ones moved by ϕ are the same as those moved by ϕ^{-1}. We will only show that if a factor is moved by ϕ, then it is moved by ϕ^{-1}, as the reverse implication is similar. Let T be a factor of trailing ones in $\lfloor d \rfloor$. After T moves when applying ϕ, it either becomes a block itself or merges with B' where B is the block which was to its right. In the first case, T is clearly a block of ones in $\lceil e \rceil = \phi(\lfloor d \rfloor)$ and so also in $\lceil e \rceil^R$. Thus, it will be moved when computing $\phi^{-1}(\lceil e \rceil)$. In the second case, it suffices to show that the leftmost entry d_i of B becomes $e_i \geq 2$ in $\lceil e \rceil$, since then T becomes a factor of trailing ones in $\lceil e \rceil^R$. If $\#B = 1$, then $e_i = d_i \geq 2$, since, otherwise, $d_i = 1$ would have been one of the trailing ones of the original block and moved to the right. On the other hand, if $\#B \geq 2$, then by P2, we have $e_i = d_i + 1 \geq 2$, which is again what we wished to show and completes the second case of the claim.

Because of the claim, ϕ^{-1} acts as a step-by-step inverse of ϕ. Indeed, moving factors right in $\lfloor d \rfloor$ corresponds to moving the same factors left in $\lceil e \rceil$. And this is equivalent to moving them right in $\lceil e \rceil^R$ by applying ϕ, while the final reversal brings the factor back to its original position. Also, what P2 does to the two ends of the remains B' of a block B are inverses of each other. This shows that B' will also be restored to itself by ϕ^{-1}, so that this map does indeed undo what was done by ϕ. This completes the proof. □

2.2. Bijection Φ for Fences

We will now show how we can use the bijection ϕ for gates to construct the desired bijection Φ for fences. We chose this path, because ϕ is simpler to describe than Φ and yet captures the most important movement of elements in the algorithm, which is on the descending segments.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s) \models n-1$ with s odd. Write $s = 2\ell+1$, where $\ell \geq 0$. Our algorithm will be simplest to state using somewhat different parameters for the corresponding fence $F = F(\beta)$. These constants first appeared in the work of Elizalde et al. [8] concerning rowmotion on fences. Call the elements of F which appear on two segments *shared* and all other elements *unshared*. It will be convenient to use different notation and conventions for ascending and descending segments. Let the ascending segments of F be $A_1, A_2, \ldots, A_{\ell+1}$ from left to right, and similarly, let $D_1, D_2, \ldots, D_\ell$ be the descending segments. Let

$$\delta_i = 1 + (\text{the number of unshared elements on } D_i) \tag{1}$$

for $1 \leq i \leq \ell$. Thus, $\delta_i = \beta_{2i}$. Similarly, let

$$\alpha_i = 1 + (\text{the number of unshared elements on } A_i) \tag{2}$$

for $1 \leq i \leq \ell + 1$. It follows that $\alpha_i = \beta_{2i-1}$ for $2 \leq i \leq \ell$, $\alpha_1 = \beta_1 + 1$, and $\alpha_{\ell+1} = \beta_s + 1$. For $i \in [\ell + 1]$, denote by $\tilde{A}_i$ the chain consisting of the unshared elements on segment A_i. Note that $\#\tilde{A}_i = \alpha_i - 1$ and $\#D_i = \delta_i + 1$, and that each element from F appears in exactly one of the $\tilde{A}_i$ or D_i.

We encode ideals I of $F(\beta)$ by pairs of sequences $a : a_1, a_2, \ldots, a_{\ell+1}$ and $d : d_1, d_2, \ldots, d_\ell$, where $a_i = \#(I \cap \tilde{A}_i)$ and $d_i = \#(I \cap D_i)$ for all i. It is

sometimes convenient to visualize these sequences as placed one above the other, with entries interlaced, that is

$$\left\lfloor \begin{matrix} a \\ d \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} a_1 & & a_2 & & \cdots & & a_\ell & & a_{\ell+1} \\ & d_1 & & d_2 & & \cdots & & d_\ell & \end{matrix} \right\rfloor.$$

Similarly, we encode filters U of $F(\beta)$ by pairs of sequences $b : b_1, b_2, \ldots, b_{\ell+1}$ and $e : e_1, e_2, \ldots, e_\ell$, where $b_i = \#(U \cap \tilde{A}_i)$ and $e_i = \#(U \cap D_i)$ for all i, and we write

$$\left\lceil \begin{matrix} b \\ e \end{matrix} \right\rceil = \left\lceil \begin{matrix} b_1 & & b_2 & & \cdots & & b_\ell & & b_{\ell+1} \\ & e_1 & & e_2 & & \cdots & & e_\ell & \end{matrix} \right\rceil.$$

A pair of sequences $\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor$ as above encodes an ideal of $F(\beta)$ if and only if the following conditions hold:

IF1 for $i \in [\ell+1]$, we have $0 \leq a_i \leq \alpha_i - 1$,
IF2 for $i \in [\ell]$, we have $0 \leq d_i \leq \delta_i + 1$,
IF3 for $i \in [\ell]$: if $d_i = \delta_i + 1$, then $a_i = \alpha_i - 1$, and if $i > 1$, then $d_{i-1} > 0$, as well,
IF4 for $i \in [\ell]$: if $a_{i+1} > 0$, then $d_i > 0$.

Note that the size of the ideal is $\sum_i a_i + \sum_i d_i$.

To obtain the conditions for $\left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil$ to encode a filter of $F(\beta)$, note that this happens if and only if

$$\left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil^R \stackrel{\text{def}}{=} \left\lfloor \begin{smallmatrix} b^R \\ e^R \end{smallmatrix} \right\rfloor \tag{3}$$

encodes an ideal of $F(\beta^R)$. Similarly, define

$$\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor^R = \left\lceil \begin{smallmatrix} a^R \\ d^R \end{smallmatrix} \right\rceil. \tag{4}$$

The following is equivalent to $\left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil$ being a filter of $F(\beta)$:

UF1 for $i \in [\ell+1]$, we have $0 \leq b_i \leq \alpha_i - 1$,
UF2 for $i \in [\ell]$, we have $0 \leq e_i \leq \delta_i + 1$,
UF3 for $i \in [\ell]$: if $e_i = \delta_i + 1$, then $b_{i+1} = \alpha_{i+1} - 1$, and if $i < \ell$, then $e_{i+1} > 0$, as well,
UF4 for $i \in [\ell]$: if $b_i > 0$, then $e_i > 0$.

Next, we define a bijection $\Phi : \mathcal{I}_k(F) \to \mathcal{U}_k(F)$, where

$$k \leq \min\{\beta_1, \beta_s\} = \min\{\alpha_1, \alpha_{\ell+1}\} - 1.$$

Given an ideal of $\mathcal{I}_k(F)$ encoded by a pair of sequences $\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor$, we apply the following steps, where we use $x := y$ to mean that x is to be assigned the value y.

PH1 For every $i \in [\ell]$, such that $d_i = 1$ and $a_{i+1} < \alpha_{i+1} - 1$, let $d_i := 0$ and $a_{i+1} := a_{i+1} + 1$.
PH2 Decompose $d : d_1, d_2, \ldots, d_\ell$ into factors by splitting between d_{i-1} and d_i for each $i \in [2, \ell]$, such that $a_i < \alpha_i - 1$. Apply ϕ (defined by P1–P2) to each factor to obtain a sequence e. Let $b := a$.
PH3 For every $i \in [\ell]$, such that $e_i = 0$ and $b_i > 0$, let $e_i := 1$ and $b_i := b_i - 1$.

Define

$$\Phi(\lfloor {a \atop d} \rfloor) = \lceil {b \atop e} \rceil.$$

For example, let $F = F(6,2,1,2,3,1,6)$ be the fence in Fig. 4, which has $\alpha : 7,1,3,7$ and $\delta : 2,2,1$. Label its elements $x_1, x_2, \ldots, x_{22}$ from left to right, and consider the ideal $I = \{x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}\} \in \mathcal{I}(F)$, which is encoded by

$$\left\lfloor {a \atop d} \right\rfloor = \left\lfloor \begin{matrix} 0 & & 0 & & 1 & & 0 \\ & 1 & & 3 & & 1 & \end{matrix} \right\rfloor.$$

In the top of Fig. 4, the elements of I are circled. Applying PH1 yields

$$\begin{matrix} 0 & & 0 & & 1 & & 1 \\ & 1 & & 3 & & 0 & \end{matrix}.$$

In step PH2, the sequence $1,3,0$ is split into two factors: the first factor is $1,3$ and the second factor is 0. Applying ϕ to each one, we get

$$\begin{matrix} 0 & & 0 & & 1 & & 1 \\ & 2 & & 2 & & 0 & \end{matrix}.$$

Finally, applying PH3 yields

$$\left\lceil {b \atop e} \right\rceil = \left\lceil \begin{matrix} 0 & & 0 & & 0 & & 1 \\ & 2 & & 2 & & 1 & \end{matrix} \right\rceil,$$

which encodes the filter $U = \{x_7, x_8, x_{10}, x_{11}, x_{15}, x_{21}\} \in \mathcal{I}(F)$, depicted in the bottom of Fig. 4.

We now prove the main theorem of this section.

Theorem 2.2. *Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ where $s = 2\ell + 1$ and*

$$k \leq \min\{\beta_1, \beta_s\}. \tag{5}$$

The map $\Phi : \mathcal{I}_k(\beta) \to \mathcal{U}_k(\beta)$ defined by PH1–PH3 *is a bijection.*

Proof. We maintain the notation established in the lead up to this theorem. To show that Φ is well defined, we need to first demonstrate that ϕ can be applied to the factors determined by PH2 in that they satisfy I1–I3. The first two conditions follow directly from the fact that $I = \lfloor {a \atop d} \rfloor$ is an ideal. For I3, we begin with d_1 in the first factor and assume, towards a contradiction, that $d_1 = \delta_1 + 1$. But then, IF3 forces $a_1 = \alpha_1 - 1$. Therefore

$$k = \#I \geq a_1 + d_1 \geq \alpha_1 = \beta_1 + 1,$$

which contradicts (5). Now, consider d_ℓ in the last factor and suppose, again towards a contradiction, that $d_\ell = 1$ when ϕ is about to be applied. Note that we must also have $a_{\ell+1} < \alpha_{\ell+1} - 1$, since otherwise we would again contradict (5) similarly to our first case. But under these conditions, PH1 would have set d_ℓ to 0, which is again a contradiction. To finish verifying I3, we must consider the splits between d_{i-1} and d_i for $2 \leq i \leq \ell$, which occur when $a_i < \alpha_i - 1$ in PH2. If $d_{i-1} = 1$, then we must have had $a_i < \alpha_i - 1$ to start with, since a_i can only increase in value, so PH1 would have again set

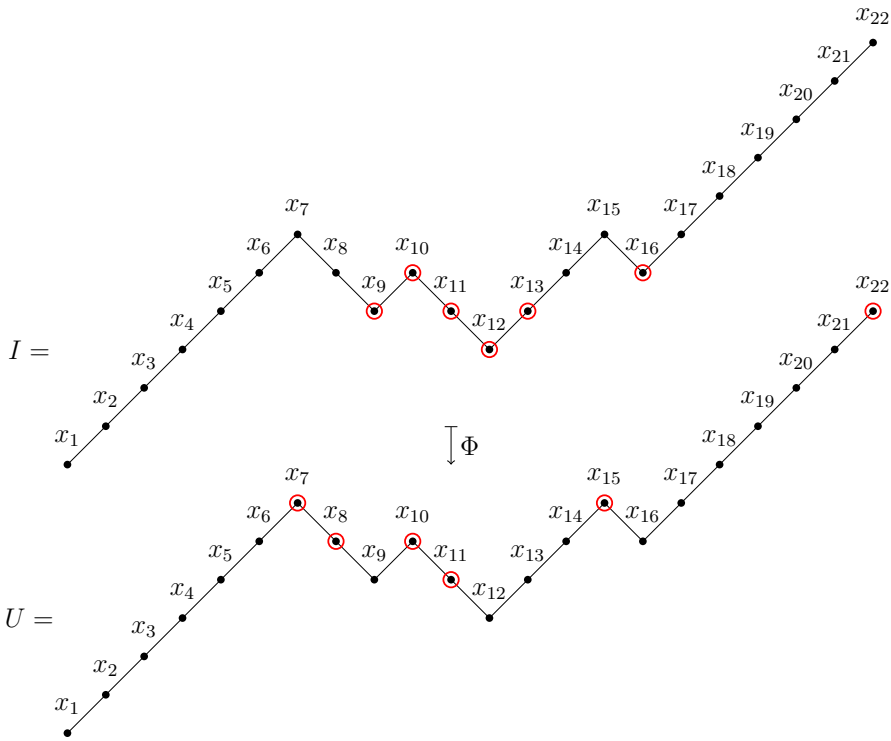

FIGURE 4. Computing $\Phi(\{x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}\})$ in $F(6, 2, 1, 2, 3, 1, 6)$

d_{i-1} to 0. If $d_i = \delta_i + 1$, then IF3 forces $a_i = \alpha_i - 1$, which contradicts the assumption in PH2. Therefore, in all cases, ϕ can be applied.

That Φ preserves cardinality follows from the fact that ϕ does and that the assignments in steps PH1 and PH3 keep the sum of the sequences equal. Therefore, to finish the proof that Φ is well defined, we need to show that $\Phi(\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor) = \left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil$ satisfies UF1–UF4. The first two items follow by the equalities and bounds imposed in PH1 and PH3 before reassignment, and from the fact that the image of ϕ satisfies U1. To check UF3, suppose $e_i = \delta_i + 1$. The "and if" clause is true, because applying ϕ gives a sequence satisfying U2. For the first clause, we will see that having $b_{i+1} < \alpha_{i+1} - 1$ leads to a contradiction. Note that the value of b_{i+1} could not have been lowered in PH3, since $e_i = \delta_i + 1 \neq 0$. Therefore, we have $a_{i+1} = b_{i+1} < \alpha_{i+1} - 1$, and the condition in PH2 forces e_i to be the end of a factor. However, since ϕ maps to restricted filters, we have that $e_i \leq \delta_i$ by U3, which is the desired contradiction. Finally, we tackle UF4 by contradiction again, assuming $b_i > 0$ and $e_i = 0$. If this had been the case, then e_i would have been reassigned to be 1 in PH3. This completes the verification that Φ is well defined.

As with ϕ, we define Φ^{-1} to be

$$\Phi^{-1}(\lceil {b \atop e} \rceil) = \Phi(\lceil {b \atop e} \rceil^R)^R. \tag{6}$$

As in the demonstration of Theorem 2.1, the proof that Φ^{-1} is well defined follows from the fact that Φ is.

We first prove that $\Phi^{-1} \circ \Phi$ is the identity. We first need to show that if $\Phi(\lfloor {a \atop d} \rfloor) = \lceil {b \atop e} \rceil$, then d gets broken into factors when applying Φ at the same indices as e^R when applying Φ^{-1}. We will show that every break point of d becomes a break point of e^R, again leaving the reverse implication to the reader. If there was a break between d_{i-1} and d_i in applying Φ, then we must have $a_i < \alpha_i - 1$ in step PH2. After applying PH3, we have $b_i \leq a_i < \alpha_i - 1$. Next, PH1 is applied to $\lceil {b \atop e} \rceil^R$ as the first step of Φ^{-1}. If b_i does not change at this step, then PH2 will still split e^R between e_i and e_{i-1} because of the previous inequality. If b_i does increase during PH1, then it must have been, because $e_i = 1$ at this stage. However, d_i was first in its factor before applying ϕ, and so, by U3, we had $e_i \neq 1$ after PH2 was applied as part of Φ. Therefore, the only way to have $e_i = 1$ at the end of PH3 is if we also decreased b_i by one in that step. In this case, $b_i < a_i < \alpha_i - 1$, which makes $b_i < \alpha_i - 1$ after adding one in PH1. Therefore, PH2 will still break at the same spot.

It is now easy to see that Φ^{-1} will act as a step-by-step inverse for Φ. Indeed, applying PH1 for Φ^{-1} undoes what PH3 did for Φ. By what we proved in the previous paragraph and the definition of ϕ^{-1}, the steps PH2 for Φ and Φ^{-1} cancel each other out. And finally, step PH3 for Φ^{-1} cancels out PH1 in Φ.

To complete the proof, we show that $\Phi^{-1} \circ \Phi$ is the identity map. This follows from Eq. (6) and the fact that $\Phi^{-1} \circ \Phi$ is the identity, since:

$$\Phi(\Phi^{-1}(\lceil {b \atop e} \rceil)) = \Phi(\Phi(\lceil {b \atop e} \rceil^R)^R) = \Phi^{-1}(\Phi(\lceil {b \atop e} \rceil^R))^R = (\lceil {b \atop e} \rceil^R)^R = \lceil {b \atop e} \rceil$$

as desired. □

3. Proof of Symmetry for Circular Fences

We will now show how slight modifications of ϕ and Φ can be used to give a bijective proof of Theorem 1.4. We use the notation

$$\overline{\mathcal{I}}(\beta) = \{I \mid I \text{ is a lower order ideal of } \overline{F}(\beta)\}$$

and

$$\overline{\mathcal{U}}(\beta) = \{U \mid U \text{ is an upper order ideal of } \overline{F}(\beta)\}.$$

Our goal is to construct a cardinality-preserving bijection $\overline{\Phi} : \overline{\mathcal{I}}(\beta) \to \overline{\mathcal{U}}(\beta)$. As before, we start with the case where ascending segments have length one.

3.1. Bijection $\overline{\phi}$ for Narrow Circular Fences

We call a circular fence $\overline{F}(\beta)$ *narrow* if its composition has the form $\beta = (1, \delta_1, 1, \delta_2, \ldots, 1, \delta_\ell)$. Let D_i be the descending segment of length δ_i. Any $I \in \overline{\mathcal{I}}(\beta)$ can be expressed as $I = \lfloor d_1, d_2, \ldots, d_\ell \rfloor$, where $d_i = \#(I \cap D_i)$ for $i \in [\ell]$, satisfying the following conditions:

ICN1 (existence) for $i \in [\ell]$, we have $0 \leq d_i \leq \delta_i + 1$,
ICN2 (ideal) for $i \in [\ell]$: if $d_i = \delta_i + 1$, then $d_{i-1} > 0$, where subscripts are taken modulo ℓ.

Similarly, the conditions for filters $U = \lceil e_1, e_2, \ldots, e_\ell \rceil$ of $\overline{F}(\beta)$ are as follows:

UCN1 (existence) for $i \in [\ell]$, we have $0 \leq e_i \leq \delta_i + 1$,
UCN2 (filter) for $i \in [\ell]$: if $e_i = \delta_i + 1$, then $e_{i+1} > 0$, where subscripts are taken modulo ℓ.

To define $\overline{\phi}$, it will be useful to define a *circular sequence* $\langle d \rangle$: $\langle d_1, d_2, \ldots, d_\ell \rangle$, which is obtained from the linear sequence $d : d_1, d_2, \ldots, d_\ell$ by considering d_ℓ as followed by d_1. Equivalently, the subscripts in a circular sequence are to be treated modulo ℓ and this will be our convention in all definitions pertaining to circular sequences. Note our calling ordinary sequences *linear* to distinguish them from the circular case.

A *factor* of $\langle d \rangle$ is a subsequence of the form $d_i, d_{i+1}, \ldots, d_j$. Note that this is a linear sequence, even though it may wrap around to the beginning of d. Call d *positive* if all its elements are positive. If d is not positive (and so has at least one zero), then a *block* B of $\langle d \rangle$ is a maximal factor of positive elements. For example, the circular sequence

$$\langle d \rangle = \langle 7, 1, 1, 0, 5, 1, 0, 0, 3 \rangle$$

has blocks $5, 1$ and $3, 7, 1, 1$. Now, the *trailing ones* of a block are defined exactly as in the linear case. Conveniently, for circular sequences, every factor of trailing ones is followed by a zero, which is why we do not need the notion of restriction for ideals in circular fences. In our example, block $5, 1$ has one trailing one and block $3, 7, 1, 1$ has two.

Now, suppose we are given $I = \lfloor d_1, d_2, \ldots, d_\ell \rfloor \in \overline{\mathcal{I}}(\beta)$ with $\overline{F}(\beta)$ narrow. If $d : d_1, d_2, \ldots, d_\ell$ is not positive, then we define $\overline{\phi}(I)$ by applying P1 and P2 for ϕ to $\langle d_1, d_2, \ldots, d_\ell \rangle$. Note that this is well defined, since factors of a circular permutation are linear. Returning to our example, we have

$$I = \lfloor 7, 1, 1, 0, 5, 1, 0, 0, 3 \rfloor \overset{\text{P1}}{\mapsto} \langle 7, 0, 1, 1, 5, 0, 1, 0, 3 \rangle \overset{\text{P2}}{\mapsto} \lceil 6, 0, 1, 1, 5, 0, 1, 4 \rceil = \overline{\phi}(I).$$

If d is positive, then we let

$$\overline{\phi} \lfloor d_1, d_2, \ldots, d_\ell \rfloor = \lceil d_1, d_2, \ldots, d_\ell \rceil.$$

The proof of the next result is very similar to that of Theorem 2.1, so the demonstration is omitted.

Theorem 3.1. *Let $\beta = (1, \delta_1, 1, \delta_2, \ldots, 1, \delta_\ell)$. The map $\overline{\phi} : \overline{\mathcal{I}}(\beta) \to \overline{\mathcal{U}}(\beta)$ defined above is a cardinality-preserving bijection.* □

3.2. Bijection $\overline{\Phi}$ for Circular Fences

Now, consider an arbitrary circular fence $\overline{F} = \overline{F}(\beta)$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell}) \models n$. We again use A_i and D_i to denote the corresponding ascending and descending segments, noting that now there are only ℓ ascending segments. Define δ_i and α_i using Eqs. (1) and (2), respectively. We now have $\delta_i = \beta_{2i}$ and $\alpha_i = \beta_{2i-1}$ for all i (unlike the linear case, there are no exceptions).

Given $I \in \overline{\mathcal{I}}(\beta)$, we continue to let $a_i = \#(I \cap \tilde{A}_i)$ and $d_i = \#(I \cap D_i)$, where $\tilde{A}_i$ also retains its previous meaning. The notation for I will be

$$\left\lfloor \begin{matrix} a \\ d \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} a_1 & & a_2 & & \cdots & & a_\ell & & a_1 \\ & d_1 & & d_2 & & \cdots & & d_\ell & \end{matrix} \right\rfloor. \tag{7}$$

Note the repetition of a_1 in the top line, which will make our future definitions simpler. The encoding for filters is changed mutatis mutandis.

We can now easily write down the conditions for being an ideal of a circular fence in terms of the a_i and d_i (all subscripts are modulo ℓ):

IC1 for $i \in [\ell]$, we have $0 \le a_i \le \alpha_i - 1$,
IC2 for $i \in [\ell]$, we have $0 \le d_i \le \delta_i + 1$,
IC3 for $i \in [\ell]$: if $d_i = \delta_i + 1$, then $a_i = \alpha_i - 1$ and $d_{i-1} > 0$,
IC4 for $i \in [\ell]$: if $a_i > 0$, then $d_{i-1} > 0$.

Similarly, $\left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil$ being a filter is equivalent to the following conditions:

UC1 for $i \in [\ell]$, we have $0 \le b_i \le \alpha_i - 1$,
UC2 for $i \in [\ell]$, we have $0 \le e_i \le \delta_i + 1$,
UC3 for $i \in [\ell]$: if $e_i = \delta_i + 1$, then $b_{i+1} = \alpha_{i+1} - 1$ and $e_{i+1} > 0$,
UC4 for $i \in [\ell]$: if $b_i > 0$, then $e_i > 0$.

We now modify PH1–PH3 for the circular case. Given $\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor$ as in (7), perform the following operations. In all steps, the indices are taken modulo ℓ.

PHC1 For every $i \in [\ell]$, such that $d_i = 1$ and $a_{i+1} < \alpha_{i+1} - 1$, let $d_i := 0$ and $a_{i+1} := a_{i+1} + 1$.
PHC2 If there exists some index $i \in [\ell]$ with $a_i < \alpha_i - 1$, then split $\langle d \rangle$ into factors between d_{i-1} and d_i for each such i and apply ϕ to each factor. If no such i exists, then compute $\overline{\phi}(\lfloor d \rfloor)$. In both cases, let e be the resulting sequence. Let $b := a$.
PHC3 For every $i \in [\ell]$, such that $e_i = 0$ and $b_i > 0$, let $e_i := 1$ and $b_i := b_i - 1$.

Define

$$\overline{\Phi}(\left\lfloor \begin{smallmatrix} a \\ d \end{smallmatrix} \right\rfloor) = \left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil.$$

Let us look at two examples which will illustrate the two cases in step PHC2. First, consider the circular fence $\overline{F}(\beta)$ where $\beta = (2, 1, 2, 3, 1, 2, 2, 1)$, so

$$\alpha : 2, 2, 1, 2 \quad \text{and} \quad \delta : 1, 3, 2, 1,$$

as illustrated in Fig. 5. Let $I = \{x_1, x_2, x_3, x_4, x_5, x_9, x_{12}\} \in \overline{\mathcal{I}}(\beta)$, which is encoded by

$$\left\lfloor \begin{matrix} a \\ d \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} 1 & & 1 & & 0 & & 0 & & 1 \\ & 2 & & 1 & & 1 & & 1 & \end{matrix} \right\rfloor.$$

This ideal is indicated by the circled nodes in the top poset in Fig. 5. Applying PHC1 yields

$$\begin{matrix} 1 & & 1 & & 0 & & 1 & & 1 \\ & 2 & & 1 & & 0 & & 1 & \end{matrix}.$$

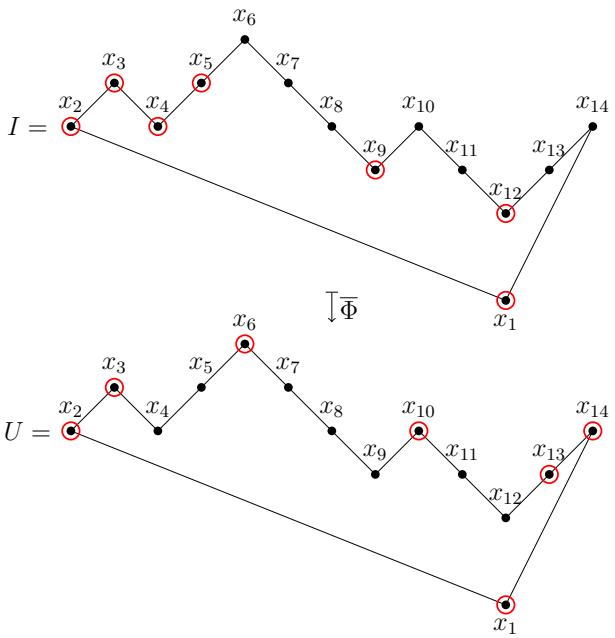

FIGURE 5. Computing $\overline{\Phi}(\{x_1, x_2, x_3, x_4, x_5, x_9, x_{12}\})$ in $\overline{F}(2, 1, 2, 3, 1, 2, 2, 1)$

In step PHC2, since $a_i = \alpha_i - 1$ for all i, we simply apply $\overline{\phi}$ to the sequence $\lfloor d \rfloor = \lfloor 2, 1, 0, 1 \rfloor$, which gives $\lceil e \rceil = \lceil 1, 0, 1, 2 \rceil$. Finally, applying PHC3 to

$$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \end{array}$$

yields

$$\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 2 \end{bmatrix},$$

which encodes the filter $U = \{x_1, x_2, x_3, x_6, x_{10}, x_{13}, x_{14}\} \in \overline{\mathcal{U}}(\beta)$, as illustrated in the bottom poset in Fig. 5.

If, instead, we apply $\overline{\Phi}$ to the ideal $I = \{x_1, x_2, x_3, x_4, x_9, x_{12}\} \in \overline{\mathcal{I}}(\beta)$, which is encoded by

$$\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ & 2 & 1 & 1 & 1 \end{bmatrix},$$

step PHC1 yields

$$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ & 2 & 1 & 0 & 1 \end{array}.$$

Now, in step PHC2, we have $a_2 = 0 < 1 = \alpha_2 - 1$, so we split $\langle d \rangle$ between $d_1 = 2$ and $d_2 = 1$. Applying ϕ to the resulting linear factor $1, 0, 1, 2$, we obtain

the sequence $0, 1, 2, 1$, and so $e : 1, 0, 1, 2$. Finally, applying PHC3 to

$$\begin{matrix} 1 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \end{matrix}$$

does not produce any change, and so

$$\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \end{bmatrix},$$

which encodes the filter $U = \{x_1, x_2, x_3, x_{10}, x_{13}, x_{14}\} \in \overline{\mathcal{U}}(\beta)$.

To prove that $\overline{\Phi}$ is bijective, we will use the definition of reversal for ideals given by (4), remembering that for circular fences a_1 appears twice in a. Therefore

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_\ell & a_1 \\ & d_1 & d_2 & \cdots & d_\ell \end{bmatrix}^R = \begin{bmatrix} a_1 & a_\ell & \cdots & a_2 & a_1 \\ & d_\ell & d_{\ell-1} & \cdots & d_1 \end{bmatrix}.$$

Similarly, reversal for filters is given by (3).

Theorem 3.2. *Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell})$. The map $\overline{\Phi} : \overline{\mathcal{I}}(\beta) \to \overline{\mathcal{U}}(\beta)$ defined by* PHC1–PHC3 *is a cardinality-preserving bijection.*

Proof. We will use the notation we have established above. If there is an index i in step PHC2 with $a_i < \alpha_i - 1$, then this map is very similar to Φ. The proof in this case essentially follows the lines of that of Theorem 2.2, and so we omit the details.

Assume now that, in step PHC2, we have $a_i = \alpha_i - 1$ for all i. There are two possibilities depending on whether d is positive or not. First, consider what happens if d is positive. In this case $\overline{\phi}(\lfloor d \rfloor) = \lceil d \rceil$, so that in step PHC2, we have $e := d$. Since d does not change, PHC3 will undo what was done in PHC1, so that $b = a$. Thus, in this case, $\overline{\Phi}$ is the identity map at the level of encodings. It is now easy to check that this map is well defined, and trivial that it is a bijection.

Now, suppose that d is not positive. Clearly, $\overline{\Phi}$ preserves cardinality, because so does $\overline{\phi}$, and in the first and last steps, the changes take place in pairs, with one element increasing by one and the other decreasing by the same amount. We need to show that Φ is well defined in that $\left\lceil \begin{smallmatrix} b \\ e \end{smallmatrix} \right\rceil \in \overline{\mathcal{U}}(\beta)$. Therefore, we need to check UC1–UC4.

Conditions UC1 and UC2 are true because of the bounds and equalities which must be satisfied in steps PHC1 and PHC3 before making the assignments, and because in PHC2, we know that $\overline{\phi}(\lfloor d \rfloor)$ satisfies UCN2. To check UC3, we assume $e_i = \delta_i + 1$ at the end of PHC3, and thus also at the end of PHC2. Since $\overline{\phi}(\lfloor d \rfloor)$ satisfies UCN2, we have that $e_{i+1} > 0$ after PHC2, and thus also after PHC3. For the other assertion in UC3 assume, towards a contradiction, that $b_{i+1} < \alpha_{i+1} - 1$. Now, the value of b_{i+1} was not changed in PHC3, because $e_{i+1} > 0$ after PHC2. Therefore, $a_{i+1} = b_{i+1} < \alpha_{i+1} - 1$, which contradicts the fact that $a_i = \alpha_i - 1$ for all i. We also handle UC4 by contradiction, assuming that $b_i > 0$ but $e_i = 0$. Under these circumstances, e_i would have been reassigned to be 1 in PHC3. Thus, we have shown that all four conditions for a filter are satisfied.

Finally, we define $\overline{\Phi}^{-1}$ by (6) with Φ replaced by $\overline{\Phi}$. The demonstration that this is well defined, and indeed, the inverse of $\overline{\Phi}$ is much the same as the proof for Φ, and so left to the reader. □

4. Comments and Open Questions

This section is devoted to some remarks and a number of open questions which we hope the reader will be interested in pursuing.

4.1. Extending the Bijections

Even though the map $\Phi : \mathcal{I}_k(\beta) \to \mathcal{U}_k(\beta)$ from Theorem 2.2 is not well defined when condition (5) does not hold, it is possible to extend it to any value of k if we restrict the map to a particular subset of ideals, namely those for which ϕ can be applied in step PH2. We continue to use the notation established at the beginning of Sect. 2.2. We say that an ideal of $F(\beta)$ encoded by $\left\lfloor {a \atop d} \right\rfloor$ is *restricted* if, in addition to IF1–IF4, it satisfies the two conditions

IF5 $d_1 \leq \delta_1$,
IF6 either $d_\ell \neq 1$ or $a_{\ell+1} < \alpha_{\ell+1} - 1$.

Note that $I = \lfloor d_1, d_2, \ldots, d_\ell \rfloor$ is a restricted ideal of the gate $G(\delta_1, \delta_2, \ldots, \delta_\ell)$ if and only if

$$I' = \left\lfloor \begin{matrix} 1 & 0 & \cdots & 0 & 1 \\ & d_1 & d_2 & \cdots & d_\ell \end{matrix} \right\rfloor$$

is a restricted ideal of the fence $F(1, \delta_1, 1, \delta_2, \ldots, \delta_\ell, 1)$. Indeed, IF5 is the first condition in I3. And since $\alpha_{\ell+1} - a_{\ell+1} = 1$, we have $a_{\ell+1} = \alpha_{\ell+1} - 1$, and so IF6 reduces to the second condition in I3.

Similarly, we say that a filter of $F(\beta)$ encoded by $\left\lceil {b \atop e} \right\rceil$ is *restricted* if, in addition to UF1–UF4, it satisfies

UF5 $e_\ell \leq \delta_\ell$,
UF6 either $e_1 \neq 1$ or $b_1 < \alpha_1 - 1$.

Denote by $\mathcal{I}_k^r(\beta)$ and $\mathcal{U}_k^r(\beta)$ the subsets of restricted ideals in $\mathcal{I}_k(\beta)$ and restricted filters in $\mathcal{U}_k(\beta)$, respectively. The reader should keep in mind that this notation refers to restricted ideals and filters in fences, not gates. If k satisfies (5), then conditions IF5–IF6 and UF5–UF6 always hold, and so, $\mathcal{I}_k^r(\beta) = \mathcal{I}_k(\beta)$ and $\mathcal{U}_k^r(\beta) = \mathcal{U}_k(\beta)$ in this case.

A slight adaptation of the proof of Theorem 2.2 demonstrates the following.

Theorem 4.1. *Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell+1})$. For any k, the map $\Phi : \mathcal{I}_k^r(\beta) \to \mathcal{U}_k^r(\beta)$ defined by PH1–PH3 is a bijection.* □

Question 4.2. *Is it possible to give an injective proof of Theorem 1.2 using a variant of Φ?*

For example, Theorem 4.1 reduces the problem of comparing the number of ideals and filters of size k to the special case of ideals and filters that fail to satisfy IF5–IF6 and UF5–UF6. Ideals that fail to satisfy IF5 (respectively,

IF6) are in bijection with ideals of the fence obtained by removing the first (respectively, last) two segments, and similarly for filters.

In a similar vein, we wonder whether it is possible to use a variant of $\overline{\Phi}$ to give an injective proof of the following conjecture of Kantarcı Oğuz and Ravichandran.

Conjecture 4.3. [11] *If $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell})$, then $\overline{r}(\beta)$ is unimodal except when $\beta = (1, k, 1, k)$ or $(k, 1, k, 1)$ for some $k \geq 1$.*

4.2. Log-Concavity

Another important property of some real sequences is log-concavity. Call $a_0, a_1, \ldots, a_n$ *log-concave* if

$$a_i^2 \geq a_{i-1}a_{i+1}$$

for all $0 < i < n$. It is well known, and easy to prove, that if a sequence contains only positive reals then log-concavity implies unimodality. It is not true that $r(\beta)$ is always log-concave, as can be seen in the example after Conjecture 1.1 where $\beta = (1, 1)$ and $r(\beta) : 1, 2, 1, 1$. It is also possible for $\overline{r}(\beta)$ to be unimodal, but not log-concave; for example, when $\beta = (1, 1, 1, 1, 1, 1)$, we have $\overline{r}(\beta) : 1, 3, 3, 4, 3, 1$. This raises the following question.

Question 4.4. *For which β are $r(\beta)$ or $\overline{r}(\beta)$ log-concave? Even if the whole sequence is not log-concave, is there a long portion of it which is?*

4.3. Chain Decompositions

In [13], McConville, Sagan, and Smyth made another conjecture which implies Theorem 1.2 but remains open. It has to do with certain chain decompositions of posets. Let $(P, \trianglelefteq)$ be a poset. If $x, y \in P$, then an *x--y chain* in P is a totally ordered subset $C : x_1 \lhd x_2 \lhd \ldots \lhd x_l$ with $x = x_1$ and $y = x_l$. Call C *saturated* if x_{i+1} covers x_i for all $1 \leq i < l$. A *chain decomposition* (CD) of P is a partition $P = \uplus_i C_i$ where the C_i are saturated chains.

Suppose now that P is ranked with rank function rk. The *center* of a saturated x–y chain C is the average

$$\operatorname{cen} C = \frac{\operatorname{rk} x + \operatorname{rk} y}{2}.$$

Let n be the maximum rank of an element of P. Call a saturated chain *symmetric* if $\operatorname{cen} C = n/2$. A *symmetric chain decomposition*, or SCD, is a chain decomposition all of whose chains are symmetric. It is easy to see that if P admits an SCD, then its rank sequence is symmetric and unimodal. Having an SCD also implies that P has the strong Sperner property as discussed in the survey article of Greene and Kleitman [10]. Greene and Kleitman also gave a famous SCD of the Boolean algebra of all subsets of a finite set [9].

There is an analogue of SCDs for top and bottom interlacing rank sequences. As in the previous paragraph, let P be ranked with maximum rank n. Call a chain decomposition of P *top centered*, or a *TCD*, if for every chain C in the decomposition, we have $\operatorname{cen} C = n/2$ or $(n + 1)/2$. Again, a simple argument shows that if P has a TCD, then its rank sequence is top interlacing.

Similarly, a *bottom-centered chain decomposition*, or *BCD*, has all chains satisfying $\operatorname{cen} C = n/2$ or $(n-1)/2$. As expected, this property implies a bottom interlacing rank sequence.

Conjecture 4.5. [13] *The lattice $L(\beta)$ admits either an SCD, BCD, or TCD consistent with Theorem 1.2.*

McConville, Sagan, and Smyth were able to prove this conjecture using modifications of the Greene–Kleitman SCD whenever β has a most three parts or is of the form $\beta = (k, 1, k, 1, \ldots, k, 1, l)$ for some $1 \leq l \leq k$. Frustratingly, there seems to be an inductive procedure which always produces a CD of the desired type for $F(\beta)$, even though it has not been possible to prove that it always works. Let P be any finite poset and let L be the corresponding distributive lattice of lower order ideals. Let $x_1, x_2, \ldots, x_n$ be a linear extension of P. Then, any subset of P can be written as an increasing sequence with respect to this extension. For example, the fence $F(2, 4, 1)$ in Fig. 1 has linear extension

$$x_7, x_8, x_6, x_5, x_4, x_1, x_2, x_3$$

which would associate the ideal $I = \{x_1, x_6, x_7, x_8\}$ with the sequence x_7, x_8, x_6, x_1. Therefore, any two subsets can now be compared using lexicographic order on their sequences. A corresponding *lexicographic chain decomposition* or *LCD* is $L = C_1 \uplus \cdots \uplus C_l$ obtained as follows. Suppose $C_1, \ldots, C_{i-1}$ have been constructed and let $L' = C_1 \uplus \cdots \uplus C_{i-1}$. We now construct $C_i : I_1 \lhd I_2 \lhd \cdots \lhd I_j$. Suppose that the smallest rank of an element of the set difference $L - L'$ is r. Choose the lexicographically smallest element of $L - L'$ having rank r to be I_1. Let I_2 be the lexicographically smallest element of $L - L'$ which covers I_1. Continue in this way until it is not possible to pick a covering element of the current ideal for C_i from $L - L'$, at which point the chain terminates. We iterate this construction until all elements of L are in a chain.

Question 4.6. [13] *Given β, is there a linear extension of $F(\beta)$ whose corresponding LCD is an SCD, BCD, or TCD of $L(\beta)$ consistent with Theorem 1.2?*

The difficulty in proving this conjecture is not that it is hard to find such a linear extension. Indeed, so many linear extensions give a CD of the desired type that it is hard to find a common feature which runs through some subset of them.

4.4. Distributive Lattices

By the Fundamental Theorem of Finite Distributive Lattices, every distributive lattice L can be obtained as the set of lower order ideals of some poset P ordered by inclusion. In this case, we write $L = L(P)$. Given what has been discussed, the following is a natural question to ask.

Question 4.7. *What conditions on a poset P imply that the rank sequence of $L(P)$ satisfies conditions on sequences such as symmetry, unimodality, and so forth? What conditions on P guarantee that $L(P)$ has an SCD, BCD, or TCD?*

4.5. Rowmotion

Fences also have connections with dynamical algebraic combinatorics. Information about this relatively new area of combinatorics can be found in the survey articles of Roby [15] or Striker [23]. Let G be a group acting on a finite set S with orbits $\mathcal{O}$. Consider a *statistic* on S, which is a map $\mathrm{st} : S \to \{0, 1, 2, \ldots\}$. Given a real constant c, we say that st is *c-mesic* if its average over any orbit $\mathcal{O}$ is c, that is

$$\frac{\mathrm{st}\,\mathcal{O}}{\#\mathcal{O}} = c,$$

where $\mathrm{st}\,\mathcal{O} = \sum_{x\in\mathcal{O}} \mathrm{st}\,x$.

Given any poset P, there is a well-studied action called *rowmotion* on $L(P)$, viewed as the set of lower order ideals of P. The generator of rowmotion is the map $\rho : L(P) \to L(P)$ defined as follows. Given $I \in L(P)$, the antichain A of its maximal elements generates an upper order ideal U. Define $\rho(I) = L(P) - U$. Elizalde et al. [8] showed that rowmotion on $L(\beta)$ has many interesting properties, but they were unable to resolve the following conjecture.

Conjecture 4.8. *Suppose $k \geq 2$ and $\beta = (k-1, k, k, \ldots, k, k-1)$ where the number of parts is odd. For $I \in L(\beta)$, define $\mathrm{st}(I) = \#I$. Then,* st *is $n/2$-mesic, where $n = \#F(\beta)$.*

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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